A NEW FORMULA FOR π

M. Hirschhorn

December 5, 1997

1. Introduction

In a recent article [1] in the Mathematical Intelligencer, a formula for π which was recently discovered **empirically** was stated, and a proof given. The object of this note is to give a slightly different proof. Indeed we show that for $|x|<1$

$$\begin{aligned} &4\sqrt{2}\left(\sum_{k\geq 0}\frac{x^{8k+1}}{8k+1}-\sum_{k\geq 0}\frac{x^{8k+5}}{8k+5}\right)-8\left(\sum_{k\geq 0}\frac{x^{8k+4}}{8k+4}+\sum_{k\geq 0}\frac{x^{8k+6}}{8k+6}\right)\\ &\qquad=\ln\frac{(1+\sqrt{2}x+x^2)(1-x^2)^2}{(1-\sqrt{2}x+x^2)(1+x^4)}+2\tan^{-1}\frac{\sqrt{2}x}{1-x^2}+2\tan^{-1}x^2.\end{aligned}$$

If we set $x=\frac{1}{\sqrt{2}}$ we obtain the formula alluded to in the title, namely

$$\pi=\sum_{k\geq 0}\frac{1}{16^k}\left(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right).$$

2. The proof

We have, for $|x|<1$,

$$\int_0^x\frac{1}{1-u}du=\ln\frac{1}{1-x},\tag{1a}$$

$$\int_0^x\frac{1}{1+u}du=\ln(1+x),\tag{1b}$$

$$\int_0^x\frac{1}{1-u^2}du=\int_0^x\frac{1}{2}\left(\frac{1}{1-u}+\frac{1}{1+u}\right)du\tag{2a}$$

$$= \frac{1}{2}\left(\ln\frac{1}{1-x} + \ln(1+x)\right)$$

$$= \frac{1}{2}\ln\frac{1+x}{1-x} \qquad \text{by (1a),(1b),}$$

$$\text{(2b)} \qquad \int_0^x \frac{u}{1-u^2}du = \int_0^x \frac{1}{2}\left(\frac{1}{1-u} - \frac{1}{1+u}\right)du$$

$$= \frac{1}{2}\left(\ln\frac{1}{1-x} - \ln(1+x)\right)$$

$$= \frac{1}{2}\ln\frac{1}{1-x^2} \qquad \text{by (1a),(1b),}$$

$$\text{(2c)} \qquad \int_0^x \frac{1}{1+u^2}du = \tan^{-1}x,$$

$$\text{(2d)} \qquad \int_0^x \frac{u}{1+u^2}du = \frac{1}{2}\ln(1+x^2),$$

$$\text{(4a)} \qquad \int_0^x \frac{1}{1-u^4}du = \int_0^x \frac{1}{2}\left(\frac{1}{1-u^2} + \frac{1}{1+u^2}\right)du$$

$$= \frac{1}{2}\left(\frac{1}{2}\ln\frac{1+x}{1-x} + \tan^{-1}x\right)$$

$$= \frac{1}{4}\ln\frac{1+x}{1-x} + \frac{1}{2}\tan^{-1}x \qquad \text{by (2a),(2c),}$$

$$\text{(4b)} \qquad \int_0^x \frac{u}{1-u^4}du = \int_0^x \frac{1}{2}\left(\frac{u}{1-u^2} + \frac{u}{1+u^2}\right)du$$

$$= \frac{1}{2}\left(\frac{1}{2}\ln\frac{1}{1-x^2} + \frac{1}{2}\ln(1+x^2)\right)$$

$$= \frac{1}{4}\ln\frac{1+x^2}{1-x^2} \qquad \text{by (2b),(2d),}$$

$$\text{(4c)} \qquad \int_0^x \frac{u^2}{1-u^4}du = \int_0^x \frac{1}{2}\left(\frac{1}{1-u^2} - \frac{1}{1+u^2}\right)du$$

$$= \frac{1}{2}\left(\frac{1}{2}\ln\frac{1+x}{1-x} - \tan^{-1}x\right)$$

$$= \frac{1}{4}\ln\frac{1+x}{1-x} - \frac{1}{2}\tan^{-1}x \qquad \text{by (2a),(2c),}$$

$$\text{(4d)} \qquad \int_0^x \frac{u^3}{1-u^4}du = \int_0^x \frac{1}{2}\left(\frac{u}{1-u^2} - \frac{u}{1+u^2}\right)du$$

$$= \frac{1}{2}\left(\frac{1}{2}\ln\frac{1}{1-x^2} - \frac{1}{2}\ln(1+x^2)\right)$$

$$= \frac{1}{4}\ln\frac{1}{1-x^4} \qquad \text{by (2b),(2d),}$$

(4e)

$$\begin{aligned}
\int_0^x \frac{1}{1+u^4}du &= \int_0^x \frac{1}{(1-\sqrt{2}u+u^2)(1+\sqrt{2}u+u^2)}du \\
&= \frac{1}{2}\int_0^x \frac{1}{1-\sqrt{2}u+u^2}du - \frac{1}{2\sqrt{2}}\int_0^x \frac{u}{1-\sqrt{2}u+u^2}du \\
&\quad + \frac{1}{2}\int_0^x \frac{1}{1+\sqrt{2}u+u^2}du + \frac{1}{2\sqrt{2}}\int_0^x \frac{u}{1+\sqrt{2}u+u^2}du \\
&= -\frac{1}{4\sqrt{2}}\int_0^x \frac{2u-\sqrt{2}}{1-\sqrt{2}u+u^2}du + \frac{1}{4}\int_0^x \frac{1}{1-\sqrt{2}u+u^2}du \\
&\quad + \frac{1}{4\sqrt{2}}\int_0^x \frac{2u+\sqrt{2}}{1+\sqrt{2}u+u^2}du + \frac{1}{4}\int_0^x \frac{1}{1+\sqrt{2}u+u^2}du \\
&= -\frac{1}{4\sqrt{2}}\ln(1-\sqrt{2}x+x^2) + \frac{1}{4\sqrt{2}}\ln(1+\sqrt{2}x+x^2) \\
&\quad + \frac{1}{4}\int_0^x \frac{1}{(u-\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2}du + \frac{1}{4}\int_0^x \frac{1}{(u+\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2}du \\
&= \frac{1}{4\sqrt{2}}\ln\frac{1+\sqrt{2}x+x^2}{1-\sqrt{2}x+x^2} \\
&\quad + \frac{1}{2\sqrt{2}}\left(\frac{\pi}{4}-\tan^{-1}(1-\sqrt{2}x)\right) + \frac{1}{2\sqrt{2}}\left(\tan^{-1}(1+\sqrt{2}x)-\frac{\pi}{4}\right) \\
&= \frac{1}{4\sqrt{2}}\ln\frac{1+\sqrt{2}x+x^2}{1-\sqrt{2}x+x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{1-x^2},
\end{aligned}$$

(4f)

$$\int_0^x \frac{u}{1+u^4}du = \frac{1}{2}\int_0^{x^2}\frac{1}{1+v^2}dv = \frac{1}{2}\tan^{-1}x^2,$$

(4g)

$$\begin{aligned}
\int_0^x \frac{u^2}{1+u^4}du &= \frac{1}{2\sqrt{2}}\int_0^x \frac{u}{1-\sqrt{2}u+u^2}du - \frac{1}{2\sqrt{2}}\int_0^x \frac{u}{1+\sqrt{2}u+u^2}du \\
&= \frac{1}{4\sqrt{2}}\int_0^x \frac{2u-\sqrt{2}}{1-\sqrt{2}u+u^2}du + \frac{1}{4}\int_0^x \frac{1}{1-\sqrt{2}u+u^2}du \\
&\quad - \frac{1}{4\sqrt{2}}\int_0^x \frac{2u+\sqrt{2}}{1+\sqrt{2}u+u^2}du + \frac{1}{4}\int_0^x \frac{1}{1+\sqrt{2}u+u^2}du \\
&= -\frac{1}{4\sqrt{2}}\ln\frac{1+\sqrt{2}x+x^2}{1-\sqrt{2}x+x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{1-x^2},
\end{aligned}$$

(4h)
$$\int_0^x \frac{u^3}{1+u^4}du = \frac{1}{4}\ln(1+x^4),$$

(8a)

$$\int_0^x \frac{1}{1-u^8}du = \int_0^x \frac{1}{2}\left(\frac{1}{1-u^4}+\frac{1}{1+u^4}\right)du$$
$$= \frac{1}{8}\ln\frac{1+x}{1-x}+\frac{1}{4}\tan^{-1}x+\frac{1}{8\sqrt{2}}\ln\frac{1+\sqrt{2}x+x^2}{1-\sqrt{2}x+x^2}+\frac{1}{4\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{1-x^2} \qquad \text{by (4a),(4e)},$$

(8b)

$$\int_0^x \frac{u}{1-u^8}du = \int_0^x \frac{1}{2}\left(\frac{u}{1-u^4}+\frac{u}{1+u^4}\right)du = \frac{1}{8}\ln\frac{1+x^2}{1-x^2}+\frac{1}{4}\tan^{-1}x^2 \qquad \text{by (4b),(4f)},$$

(8c)

$$\int_0^x \frac{u^2}{1-u^8}du = \int_0^x \frac{1}{2}\left(\frac{u^2}{1-u^4}+\frac{u^2}{1+u^4}\right)du$$
$$= \frac{1}{8}\ln\frac{1+x}{1-x}-\frac{1}{4}\tan^{-1}x-\frac{1}{8\sqrt{2}}\ln\frac{1+\sqrt{2}x+x^2}{1-\sqrt{2}x+x^2}+\frac{1}{4\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{1-x^2} \qquad \text{by (4c),(4g)},$$

(8d)

$$\int_0^x \frac{u^3}{1-u^8}du = \int_0^x \frac{1}{2}\left(\frac{u^3}{1-u^4}+\frac{u^3}{1+u^4}\right)du = \frac{1}{8}\ln\frac{1+x^4}{1-x^4} \qquad \text{by (4d),(4h)},$$

(8e)

$$\int_0^x \frac{u^4}{1-u^8}du = \int_0^x \frac{1}{2}\left(\frac{1}{1-u^4}-\frac{1}{1+u^4}\right)du$$
$$= \frac{1}{8}\ln\frac{1+x}{1-x}+\frac{1}{4}\tan^{-1}x-\frac{1}{8\sqrt{2}}\ln\frac{1+\sqrt{2}x+x^2}{1-\sqrt{2}x+x^2}-\frac{1}{4\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{1-x^2} \qquad \text{by (4a),(4e)},$$

(8f)

$$\int_0^x \frac{u^5}{1-u^8}du = \int_0^x \frac{1}{2}\left(\frac{u}{1-u^4}-\frac{u}{1+u^4}\right)du = \frac{1}{8}\ln\frac{1+x^2}{1-x^2}-\frac{1}{4}\tan^{-1}x^2 \qquad \text{by (4b),(4f)},$$

(8g)

$$\begin{aligned}\int_0^x \frac{u^6}{1-u^8}du &= \int_0^x \frac{1}{2}\left(\frac{u^2}{1-u^4} - \frac{u^2}{1+u^4}\right)du \\ &= \frac{1}{8}\ln\frac{1+x}{1-x} - \frac{1}{4}\tan^{-1}x + \frac{1}{8\sqrt{2}}\ln\frac{1+\sqrt{2}x+x^2}{1-\sqrt{2}x+x^2} - \frac{1}{4\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{1-x^2} \qquad \text{by (4c),(4g),}\end{aligned}$$

and

(8h)

$$\int_0^x \frac{u^7}{1-u^8}du = \int_0^x \frac{1}{2}\left(\frac{u^3}{1-u^4} - \frac{u^3}{1+u^4}\right)du = \frac{1}{8}\ln\frac{1}{1-x^8} \qquad \text{by (4d),(4h).}$$

The result now follows, since for each r

$$\sum_{k\geq 0}\frac{x^{8k+r}}{8k+r} = \int_0^x \frac{u^{r-1}}{1-u^8}du.$$

Reference

[**1**] D. H. Bailey, J. M. Borwein, P. B. Borwein and S. Plouffe, The Quest for Pi, The Mathematical Intelligencer **19**(1)(1997), 50–57.