Approximations to π via the Dedekind eta function

J.M. Borwein
Math Dept., Simon Fraser Univ., Burnaby, B.C, V5A 1S6 Canada

F.G. Garvan
Math. Dept., University of Florida, Gainesville, FL 32611

March 27, 1996

Abstract. Arguably the most efficient algorithm currently known for the extended precision calculation of π is a quartic iteration due to J.M. and P.B. Borwein. In their paper, the Borwein's show how this iteration and others are intimately connected to the work of Ramanujan. This connection is shown utilizing their *alpha*-function which is defined in terms of theta-functions. They are able to find p-th order iterations based on this function using modular equations for the theta-functions. In this paper we construct an infinite family of functions α_p. Each α_p gives rise to a p-th order iteration. For $p = 4$ we obtain a quartic iteration due to the Borweins but not the one that comes from the *alpha*-function. For $p = 3$ we obtain a cubic iteration due to the Borweins that does not come from the *alpha*-function. For $p = 7$ we find a septic iteration that is analogous to the cubic iteration. For $p = 9$ we obtain a nonic (ninth order) iteration that does not seem to come from iterating the cubic twice. Our method depends on using the computer and a symbolic algebra package to find and solve certain modular equations.

1 Introduction

In Bailey, Borwein, and Borwein's paper [8] an overview of a method is given for constructing series and algorithms for finding rapid approximations for π. Although Ramanujan did not know these algorithms many of the key ingredients are in his notebooks [13]. The algorithms depend crucially on the solvable forms of certain modular equations for the theta-functions due to Ramanujan. In [8] two algorithms are given – one quartic and one quintic algorithm. In a related paper [5] a septic algorithm is sketched. In [4] a general method is given for constructing p-th order algorithms. These algorithms involve defining a sequence $\{\alpha^{(n)}\}_{n=1}^{\infty}$ recursively and for which $\alpha^{(n)}$ converges to $1/\pi$ to high order. In general, for us, p-th order convergence of a sequence $\{\alpha^{(n)}\}_{n=1}^{\infty}$ to $\alpha^{(\infty)}$ means that $\alpha^{(n)}$ tends to $\alpha^{(\infty)}$ and that

$$|\alpha^{(n+1)} - \alpha^{(\infty)}| \le C|\alpha^{(n)} - \alpha^{(\infty)}|^p \tag{1.1}$$

for some constant $C > 0$. The proof of p-th order convergence depends crucially on identifying $\alpha^{(n)}$ as the value of a certain function $\alpha(\cdot)$, which can be defined in terms of elliptic integrals or equivalently in terms of theta-functions.

Since [8] was first written, Borwein and Borwein [6] found an amazing cubic algorithm. See also [7]. This algorithm comes from a certain hypergeometric analog of elliptic integrals that was studied by Ramanujan. In this paper, we make an attempt to unify some of these results and find new algorithms. Instead of a fixed function $\alpha(\cdot)$, we define an infinite family of functions $\alpha_p(\cdot)$ for $p > 1$. Our goal is to construct for each p, a p-th order iteration which converges to $1/\pi$, using the function $\alpha_p(\cdot)$.

In Section 2 we briefly describe the Borwein and Borwein α-function. In Section 3 we define $\alpha_p(\cdot)$ (for each $p > 1$), in terms of Dedekind's eta function. We find that $\alpha_p(\cdot)$ satisfies a nice modular transformation property, and a nice p-th order modular equation. In Section 4 we show how the results in Section 3 may be used to construct p-th order iterations which converge to $1/\pi$. The method is illustrated with some MAPLE sessions. In Section 5 we give a brief overview of how our method relates to known quadratic, cubic and quartic Borwein and Borwein iterations. Details are given how the cases $p = 2, 4$ relate to the quadratic and quartic algorithms. In Section 6 we show how the case $p = 3$ gives the Borwein and Borwein cubic algorithm.

Our main goal in this paper is to somehow mimic the Borwein and Borwein cubic algorithm and obtain analogous higher order algorithms. In Section 7 we obtain an explicit solvable septic iteration which converges to $1/\pi$. In Section 8 we obtain an explicit solvable nonic (ninth order) iteration which converges to $1/\pi$. This nonic iteration does not appear to come from iterating the cubic twice.

Given the *organic* nature of this document, we hope, in a later version of this paper, to provide more complete details and improvements of the septic and nonic algorithms. In a later version we will also include some *mixed* order algorithms. For instance, although the case $p = 2$ leads naturally to a quadratic iteration we may instead use it to construct a new cubic iteration.

2 The function $\alpha(r)$

Before we can define the *alpha*-function we need the following classical theta functions:

$$\theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}, \tag{2.1}$$

$$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{2.2}$$

$$\theta_4(q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}. \tag{2.3}$$

The *alpha*-function can be defined as

$$\alpha(r) := \frac{\frac{1}{\pi} - 4\sqrt{r}q\frac{\dot{\theta}_4}{\theta_4}}{\theta_3^4} \qquad \text{(where } q := \exp(-\pi\sqrt{r})\text{)}. \tag{2.4}$$

As r tends to infinity we see that q tends to zero so that we have

$$\lim_{r\to\infty} \alpha(r) = \frac{1}{\pi}. \tag{2.5}$$

In [8, Theorem 3, p. 215] Borwein, Borwein and Bailey are able to express $\alpha(p^2 r)$ in terms of $\alpha(r)$ and various theta functions. Utilizing p-th order modular equations for the theta functions, they then are able to construct p-th order iterations that converge to $\frac{1}{\pi}$. In the next section we show how to construct p-th order iterations in a different way. Instead of a single *alpha*-function we construct an infinite family of α_p.

3 The function α_p

In this section we construct an infinite family of functions α_p, where p is any integer greater than 1. Theoretically it possible to find an update to any order; ie. an equation relating $\alpha_p(N^2 r)$ with $\alpha_p(r)$. We will find that this relation is particularly nice when $N = p$. This will give rise to p-th order iterations with a nice form. Our functions are constructed from the Dedekind eta function instead of the theta functions.

Let $q := \exp(2\pi i\tau)$ (with $\Im\tau > 0$). As usual the Dedekind eta function is defined as

$$\eta(\tau) := \exp(\pi i\tau) \prod_{n=1}^{\infty} (1 - \exp(2\pi i n\tau)) \tag{3.1}$$

$$= q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{3.2}$$

Then

$$\eta(-1/\tau) = \sqrt{\frac{\tau}{i}}\, \eta(\tau). \tag{3.3}$$

See [12, p. 121] for a proof. Now for $p > 1$ (a positive integer) we define

$$B_p(r) := \frac{\eta^p(\tau)}{\eta(p\tau)}, \qquad C_p(r) := \frac{\eta^p(p\tau)}{\eta(p\tau)}. \tag{3.4}$$

where $\tau = i\sqrt{r}/\sqrt{p}$ and $q = \exp(-2\pi\sqrt{r}/\sqrt{p})$. It should be noted that the functions B_3 and C_3 occured naturally in the Borwein-Borwein cubic iteration [6], [7]. Define

$$\alpha_p(r) := \frac{\left(\frac{1}{\pi} - q\frac{8\sqrt{r}}{(p-1)\sqrt{p}}\frac{\dot{B}}{B}\right)}{A_p(r)}, \tag{3.5}$$

where

$$A_p(r) := q\left(\frac{24}{p^2 - 1}\right)\left\{\frac{\dot{C}}{C} - \frac{\dot{B}}{B}\right\}. \tag{3.6}$$

Here $\dot{B} = \frac{dB}{dq}$. From (3.2) we have

$$A_p(r) = 1 + O(q), \tag{3.7}$$

and

$$A_p(1/r) = rA_p(r), \tag{3.8}$$

which follows from (3.3). The definition of α_p was chosen so that it had a form analogous to that of (2.4) and that it satisfied a transformation like (3.9) below. Using (3.3) and (3.8) it is not hard to show that

$$\alpha_p(1/r) = \frac{\frac{(p+1)}{3\sqrt{p}}\sqrt{r} - \alpha_p(r)}{r}. \tag{3.9}$$

Substituting $r = 1$ gives

$$\alpha_p(1) = \frac{p+1}{6\sqrt{p}}, \tag{3.10}$$

Since $q \to 0$ as $r \to \infty$ we see that

$$\lim_{r\to\infty} \alpha_p(r) = \frac{1}{\pi}. \tag{3.11}$$

Theorem 3.1 *thm3.1 let $N, p \geq 1$ be fixed. We have*

$$\alpha_p(N^2 r) = \alpha_p(r) m_{N,p}(r) + \sqrt{r}\epsilon_{N,p}(r), \tag{3.12}$$

where

$$\epsilon_{N,p} = \frac{p+1}{3\sqrt{p}} \left\{ \frac{q\frac{\dot{B}}{B} - Nq^N\frac{\dot{B}}{B}(q^N)}{q^N\frac{\dot{C}}{C} - q^N\frac{\dot{B}}{B}(q^N)} \right\}, \tag{3.13}$$

and

$$m_{N,p} = \frac{A_p(r)}{A_p(N^2 r)}. \tag{3.14}$$

Further

$$A_p = \frac{1}{p-1}\left[pP(q^p) - P(q)\right], \tag{3.15}$$

where

$$P(q) := 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = 24q\frac{\dot{\eta}}{\eta}. \tag{3.16}$$

Theorem 3.1 *thm3.1 The statement (3.12) follows easily from (3.5) and (3.6). Equation (3.15) follows easily from the product expansion (3.2) and the definition in (3.4).* □

When $N = p$, the function $\epsilon_{N,p}$ has a nice form

$$\epsilon_{p,p}(r) = \frac{\sqrt{p}}{3}\left(1 - m_{p,p}(r)\right), \tag{3.17}$$

so that

$$\alpha_p(p^2 r) = \alpha_p(r) m_{p,p}(r) + \frac{\sqrt{rp}}{3}\left(1 - m_{p,p}(r)\right). \tag{3.18}$$

The proof of (3.17) follows easily from (3.2), (3.4) and (3.15). From (3.8) and (3.14) we have

$$m_{p,p}(1/p) = p. \tag{3.19}$$

By using (3.18) and (3.19) we find that (3.9) and (3.18) give rise to two equations involving $\alpha_p(p)$ and $\alpha_p(1/p)$. These equations may be solved easily to yield

$$\alpha_p(1/p) = \frac{1}{3} \qquad \text{(Not a bad starting point for } 1/\pi\text{)}. \tag{3.20}$$

4 The Symbolic Search for Iterations

In this section we show how a computer algebra package like MAPLE can be used to generate p-th order iterations that converge to $1/\pi$ from the α_p. For fixed a initial value r_0 and a fixed p we define the sequence $\{\alpha_n\}_n^\infty$ by

$$\alpha_n := \alpha_p\left(r_0\, p^{2n}\right). \tag{4.1}$$

Of course, as it stands, although it is clear that α_n converges to $1/\pi$ to p-th order, it is not very practical. The idea is to write the sequence recursively. Therefore there are two problems to solve:

1. Find initial values $\alpha_p(r_0)$.
2. Get α_n in terms of α_{n-1}.

Table 1: MAPLE functions

MAPLE function	Description
`etaq(q,k,t)`	q-series expansion of $\prod_{j=1}^{\infty}(1-q^{kj})$ to order $\mathrm{O}\left(q^t\right)$
`etamake(f,t)`	η-product expansion of q-series f to order $\mathrm{O}\left(q^t\right)$
`findhom(L,n,t)`	Finds a set of homogeneous relations of degree n satisfied by the set L of q-series to order $\mathrm{O}\left(q^t\right)$

Table 1 contains some functions we defined in MAPLE and that we used to find and prove initial values and iterations.

Our function `etaq` utilizes the expansion due to Euler:

$$\prod_{j=1}^{\infty}(1-q^{kj}) = \sum_{n=-\infty}^{\infty}(-1)^n q^{3kn(n-1)/2} \tag{4.2}$$

We are then able to effectively compute q-series expansions of B_p, C_p, A_p. We have written MAPLE procedures to compute all the necessary functions.

4.1 Initial values

We illustrate a search for initial values with a MAPLE session. The file `funcs` contains all the necessary MAPLE functions.

```
>   read funcs:

>   readlib(lattice):

>   Digits := 30:

>   alpha(7,1);
```

$$.503952630678969636286022048314$$

```
>   x:=":

>   minpoly(x,2);
```

$$-16+63\_X^2$$

```
>   alpha(7,1/7);
```

$$.333333333333333333333333333334$$

```
>    x:=alpha(2,3);
```

$$x := .3236973011009816422947466284 21$$

```
>    minpoly(x,4);
```

$$-49 + 528\_X^2 - 576\_X^4$$

```
>    solve(");
```

$$\frac{\sqrt{6}}{4} + \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{4} - \frac{\sqrt{3}}{6}, -\frac{\sqrt{6}}{4} + \frac{\sqrt{3}}{6}, -\frac{\sqrt{6}}{4} - \frac{\sqrt{3}}{6}$$

```
>    evalf(alpha(2,3)-(3*sqrt(6)-2*sqrt(3))/12,100);
```

$$-.4\,10^{-99}$$

We have used MAPLE's implementation of the lattice algorithm [?] The MAPLE function `minpoly(x,n)` finds a polynomial of degree n with small integer coefficients satisfied by the real floating point number x. The output depends on the number of digits computed for the approximation x. In our session the function `alpha(p,r)` corresponds to $\alpha_p(r)$. Observe that we were able to verify (3.10) and (3.20) for $p = 7$. We found a nice quartic polynomial which appears to be satisfied by $\alpha_2(3)$ and that it appears that

$$\alpha_2(3) = \frac{3\sqrt{6} - 2\sqrt{3}}{12}. \tag{4.3}$$

Once we have such a conjectured value for $\alpha_p(r_0)$ it is possible to prove the result by computing to enough digits. In this way we were able to find many initial values. These are given in Table 2.

4.2 Modular Equations

The problem is to find a recurrence relation for the α_n, defined in (4.1). An examination of (3.18) reveals that we need to get $m_{p,p}(p^2 r)$ in terms of $m_{p,p}(r)$. From (3.14) we see that this is equivalent to finding a relation between $A_p(q)$, $A_p(q^p)$ and $A_p(q^{p^2})$. Here we have written A_p as a function of q and not r. By using (3.15), it can be shown that $A_p(q)$ is a modular form

Table 2: Initial values $\alpha_p(r_0)$

p	r_0	$\alpha_p(r_0)$
2	3	$\frac{3\sqrt{6}-2\sqrt{3}}{12}$
	4	
	5	
	6	
	7	
3	2	

of weight two on a certain congruence subgroup. This implies that all three functions $A_p(q)$, $A_p(q^p)$ and $A_p(q^{p^2})$ are modular forms of weight two on a certain congruence subgroup and hence must satisfy an algebraic relation of the form

$$P(A(q), A(q^p), A(q^{p^2})) = 0, \tag{4.4}$$

where P is a certain rational homogeneous polynomial of degree k, say. The crucial observation is that the left-hand side of (4.4) is a modular form of weight $2k$. It is well-known that

The dimension of the space of modular forms (above) of weight $2k \sim c_1\, k$

(for some nonzero constant c_1, see [10])

and

$$\text{the number of monomials } A_p^{k_1}(q)\, A_p^{k_2}(q^p) A_p^{2k-k1-k2}(q^{p^2}) \sim 2k^2$$

Hence there will always be a relation for larger enough k. Such a relation can be found and proved symbolically. This is really a *linear* problem. The q-series expansion of each monomial up to a certain power of q can be easily computed and stored as a column in a matrix. Then finding homogeneous relations is then equivalent to finding the nullspace of a certain matrix. Such relations can be proved by verifying them to a high enough power of q using the theory of modular forms. See [7] for more details.

We illustrate the case $p = 2$ with a MAPLE session.

```
>   read funcs:

>   read findhom:

>   A2:=Aseries(2);
```

$$A2 := q \longrightarrow 1 + 24q + 24q^2 + \cdots + 3744q^{99}$$

```
>   findhom([A2(q),A2(q^2),A2(q^4)],2,100);
```

# of terms, 22

-----RELATIONS-----of order---,2

$$\left\{X_1^2 - 2X_1X_2 - 7X_2^2 - 8X_2X_3 + 16X_3^2\right\}$$

We now check the relations to O(q^t)

---RELATION----, 1, ---checks to order---

$$O(q^{100})$$

The function `Aseries(2)` gives the q-series expansion of $A_2(q)$. Using our function `findhom` we found that $x = A_2(q)$, $y = A_2(q^2)$, $z = A(q^4)$ seem to satisfy the equation

$$x^2 - 2xy - 7y^2 - 8yz + 16z^2 = 0, \tag{4.5}$$

at least to $O(q^{100})$, which is enough for a proof. By solving this equation we see that

$$m_{2,2}(q^2) = \frac{4}{1 + \sqrt{(4 - m_{2,2}(q))(2 + m_{2,2}(q))}}. \tag{4.6}$$

4.3 Iteration Construction

Once we have $m_{p,p}(p^2 r)$ in terms of $m_{p,p}(r)$ constructing a p-th order iteration is a simple matter. We illustrate the construction for $p = 2$. We take $r_0 = 1/2$ as our initial value. We could have used any value from Table 2. We define two sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{m_n\}_{n=0}^{\infty}$:

$$\alpha_n := \alpha_2(2^{2n-1}), \tag{4.7}$$

$$m_n := m_{2,2}(2^{2n-1}). \tag{4.8}$$

Here we consider $m_{2,2}$ as a function of r. From (3.19) and (3.20) we have

$$\alpha_0 := \frac{1}{3}, \tag{4.9}$$

$$m_0 := 2. \tag{4.10}$$

From (3.18) and (4.6) we have for $n \geq 1$

$$m_n = \frac{4}{1 + \sqrt{(4 - m_{n-1})(2 + m_{n-1})}}, \tag{4.11}$$

$$\alpha_n = m_{n-1}\,\alpha_{n-1} + \frac{2^{n-1}}{3}(1 - m_{n-1}). \tag{4.12}$$

Equations (4.9)–(4.12) uniquely define the two sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{m_n\}_{n=0}^{\infty}$. The convergence is quadratic. We illustrate with a MAPLE session.

```
>   read iteration1:

>   it1(5,100);
```

1,	$3.00\ldots0$,	$.141593$
2,	$3.1407\ldots$,	$.838172\,10^{-3}$
3,	$3.141592646\ldots$,	$.676494\,10^{-8}$
4,	$3.14159265358979323828\ldots$,	$.175696\,10^{-18}$
5,	$3.1415926535897932384\ldots884197115\ldots$,	$.536215\,10^{-40}$

The function `it1(5,100)` computes the first 5 iterations to 100 digits and an approximation for the difference between $1/\alpha_n$ and π. We also note that this iteration is really a known quadratic iteration [6, Iteration 3.6, p. 700].

5 The Quadratic and Quartic Iterations

In the previous section we showed how to construct a quadratic iteration from the case $p = 2$. In this section we show how the cases $p = 2$ and $p = 4$ relate to the Borwein's quadratic and quartic iterations. For an overview of the method see [8]. For more complete details see [4], [6]. In this and the next section we summarise the results needed and omit the proofs. Before stating these results, we first give a brief overview.

In the previous section we saw how a quadratic iteration can be constructed using the modular equation (4.6) which relates $m_{2,2}(q^2)$ and $m_{2,2}(q)$. It is useful it introduce auxiliary functions. These functions are modular forms and are usually denoted by $a(q)$, $b(q)$ and $c(q)$. In each case, these auxiliary functions have some remarkable properties:

(1) $a(q)$ is some k-th root of $A_p(q)$

(2) There exist linear relationships between the functions $a(q)$, $b(q)$, $c(q)$, $a(q^p)$, $b(q^p)$ and $c(q^p)$.

(3) There is a simple polynomial equation relating $a(q)$, $b(q)$ and $c(q)$. In fact, for the quadratic and quartic iterations of this section and the cubic iteration of the next section this relation is precisely

$$a^p = b^p + c^p. \tag{5.1}$$

Instead of defining a sequence $\{m_n\}_{n=0}^{\infty}$ which comes from the function $m_{p,p}$ we define two auxiliary sequences $\{s_n\}_{n=1}^{\infty}$ and $\{s_n^*\}_{n=1}^{\infty}$ which will come from the functions

$$s(q) := \frac{c(q)}{a(q)}, \qquad s^*(q) := \frac{b(q)}{a(q)}. \tag{5.2}$$

In this way, (5.1) becomes

$$s^p + (s^*)^p = 1. \tag{5.3}$$

It will turn out that the multiplier $m_{p,p}$ can be written simply in terms of $s(q^p)$.

We now consider the first case $p = 2$. By [4, p.698] we have

$$A_2(q) = 2P(q^2) - P(q) = \theta_3^4(q) + \theta_2^4(q).$$

Following [4] we define

$$a(q) := \theta_3^4(q) + \theta_2^4(q), \tag{5.4}$$
$$b(q) := \theta_4^4(q), \tag{5.5}$$
$$c(q) := 2\theta_2^2(q)\theta_3^2(q). \tag{5.6}$$

Then we have the following results

$$a^2 = b^2 + c^2, \tag{5.7}$$
$$m_{2,2}(r) = \frac{a(q)}{a(q^2)}, \tag{5.8}$$
$$a(q^2) = \frac{a(q) + 3b(q)}{4}, \tag{5.9}$$
$$a(q) = a(q^2) + 3c(q^2). \tag{5.10}$$

Now we let

$$s(q) := \frac{c(q)}{a(q)}, \qquad s^*(q) := \frac{b(q)}{a(q)}. \tag{5.11}$$

Then from (5.7), (5.9)–(5.11) we have

$$1 + 3s(q^2) = 1 + 3\frac{c(q^2)}{a(q^2)} = \frac{a(q)}{a(q^2)}, \tag{5.12}$$
$$1 + 3s^*(q) = 1 + 3\frac{b(q)}{a(q)} = 4\frac{a(q^2)}{a(q)}, \tag{5.13}$$

and

$$s^2 + (s^*)^2 = 1. \tag{5.14}$$

Now, by (5.8),(5.12) and (5.13) we have

$$m_{2,2}(r) = \left[1 + 3s(q^2)\right], \tag{5.15}$$
$$\left[1 + 3s(q^2)\right]\left[1 + 3s^*(q)\right] = 4, \tag{5.16}$$

and, by (3.18), and (5.15) we have

$$\alpha_2(4r) = \alpha_2(r)m_{2,2}(r) + \frac{\sqrt{2r}}{3}(1 - m_{2,2}(r)) \tag{5.17}$$
$$= \alpha_2(r)\ [1 + 3S(4r)] - \frac{\sqrt{2r}}{3}S(4r);$$

where

$$S(r) = s(q), \qquad q = \exp(-2\pi\sqrt{r}/\sqrt{2}). \tag{5.18}$$

Now, from (3.19), (3.20) we know that

$$\alpha_2(1/2) = \frac{1}{3}, \tag{5.19}$$

$$m_{2,2}(1/2) = p = 2, \tag{5.20}$$

and so

$$2 = 1 + 3S(2),$$

$$S(2) = \frac{1}{3}. \tag{5.21}$$

By letting,

$$\alpha_n := \alpha\left(4^n \cdot \frac{1}{2}\right), \tag{5.22}$$

$$s_n := S\left(4^n \cdot \frac{1}{2}\right) = s\left(\exp(-2\pi 2^{n-1})\right), \tag{5.23}$$

$$s_n^* := S^*\left(4^n \cdot \frac{1}{2}\right), \tag{5.24}$$

we have the following

Theorem 5.1 *thm5.1 (Borwein and Borwein [6, Iteration 3.6, p. 700]) Define sequences $\{\alpha_n\}$, $\{s_n\}$, and $\{s_n^*\}$ by*

$$\alpha_0 := \alpha(1/2) = \frac{1}{3}; \tag{5.25}$$

$$s_1 := S(2) = \frac{1}{3}, \tag{5.26}$$

$$(s_n)^2 + (s_n^*)^2 = 1; \tag{5.27}$$

$$(1 + 3s_n)(1 + 3s_{n-1}^*) = 4; \tag{5.28}$$

$$\alpha_n = (1 + 3s_n)\alpha_{n-1} - 2^{n-1}s_n. \tag{5.29}$$

Then α_n converges quadratically to $\frac{1}{\pi}$.

We now consider the case $p = 4$ and show how it is related to Borwein and Borwein quartic iteration [8, Algorithm1] and how it coincides with another Borwein and Borwein quartic iteration [6, Iteration 3.4, p. 700]. From (3.15), [4, Chapter 9], [6, Theorem 2.2] we have

$$A_4(q) = 4P(q^4) - P(q) = \left[\theta_3^4(q)\right]^4. \tag{5.30}$$

Following [4] we define

$$a(q) := \theta_3(q), \tag{5.31}$$

$$b(q) := \theta_4(q), \tag{5.32}$$

$$c(q) := \theta_2(q). \tag{5.33}$$

Then we have the following results

$$a^4 = b^4 + c^4, \tag{5.34}$$

$$m_{4,4}(r) = \left[\frac{a(q)}{a(q^4)}\right]^4, \tag{5.35}$$

$$a(q^4) = \frac{a(q) + b(q)}{2}, \tag{5.36}$$

$$a(q) = a(q^4) + c(q^4). \tag{5.37}$$

As usual we let

$$s(q) := \frac{c(q)}{a(q)}, \qquad s^*(q) := \frac{b(q)}{a(q)}. \tag{5.38}$$

Then, by (5.34), (5.36)–(5.37) we have

$$1 + s(q^4) = 1 + \frac{c(q^4)}{a(q^4)} = \frac{a(q)}{a(q^4)}, \tag{5.39}$$

$$1 + s^*(q) = 1 + \frac{b(q)}{a(q)} = 2\frac{a(q^4)}{a(q)}, \tag{5.40}$$

and

$$s^4 + (s^*)^4 = 1. \tag{5.41}$$

Now, from (5.35), (5.39)–(5.41) we have

$$m_{4,4}(r) = \left[1 + s(q^4)\right]^4, \tag{5.42}$$

$$\left[1 + s(q^4)\right]\left[1 + s^*(q)\right] = 2, \tag{5.43}$$

$$s^4(q) + (s^*)^4(q) = 1, \tag{5.44}$$

and, by (3.18), (5.42) we have

$$\alpha_4(16r) = \alpha_4(r) m_{4,4}(r) + \frac{2\sqrt{r}}{3}(1 - m_{4,4}(r)) \tag{5.45}$$

where

(5.46)

$$S(r) = s(q), \qquad q = \exp(-2\pi\sqrt{r}/\sqrt{4}) = \exp(-\pi\sqrt{r}).)$$

From (3.19), (3.20) we know that

$$\alpha_4(1/4) = \frac{1}{3}, \tag{5.47}$$

$$m_{4,4}(1/3) = p = 4, \tag{5.48}$$

and so

$$4 = [1 + S(4)]^4,$$

$$S(4) = \sqrt{2} - 1. \tag{5.49}$$

By letting,

$$\alpha_n := \alpha\left(16^n \cdot \frac{1}{4}\right), \tag{5.50}$$

$$s_n := S\left(16^n \cdot \frac{1}{4}\right) = s\left(\exp(-4\pi 4^{n-1})\right), \tag{5.51}$$

$$s_n^* := S^*\left(16^n \cdot \frac{1}{4}\right), \tag{5.52}$$

we have the following

Theorem 5.2 *thm2 (Borwein and Borwein [6, Iteration 3.4, p. 700]) Define sequences* $\{\alpha_n\}$, $\{s_n\}$, *and* $\{s_n^*\}$ *by*

$$\alpha_0 := \alpha(1/4) = \frac{1}{3}; \tag{5.53}$$

$$s_1 := S(2) = \sqrt{2} - 1, \tag{5.54}$$

$$(s_n)^4 + (s_n^*)^4 = 1; \tag{5.55}$$

$$(1 + 3s_n)(1 + 3s_{n-1}^*) = 2; \tag{5.56}$$

$$\alpha_n = (1 + s_n)^4 \alpha_{n-1} + \frac{4^n}{3}(1 - (1 + s_n)^4). \tag{5.57}$$

Then α_n *converges quartically to* $\frac{1}{\pi}$.

6 The Cubic

We summarise the results of [6, 7] and how they correspond to the case $p = 3$. Define

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \tag{6.1}$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{(m-n)} q^{m^2+mn+n^2}, \qquad (\omega = \exp(2\pi i/3) \tag{6.2}$$

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2}. \tag{6.3}$$

Then

$$A_3(q) = [a(q)]^2, \tag{6.4}$$

$$b(q) = \frac{\eta^3(\tau)}{\eta(3\tau)} = B_3, \tag{6.5}$$

$$c(q) = 3\frac{\eta^3(3\tau)}{\eta(\tau)} = C_3, \tag{6.6}$$

$$m_{3,3}(r) = \left[\frac{a(q)}{a(q^3)}\right]^2. \tag{6.7}$$

We also mention the identity

$$\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} = 1 + 6\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-q^n}. \tag{6.8}$$

This equation was known to Ramanujan [3, p. 199]. See [2] for a proof using theta-functions.

We have

$$a(q^3) = \frac{a(q)+2b(q)}{3}, \tag{6.9}$$

$$a(q) = a(q^3) + 2c(q^3), \tag{6.10}$$

$$a^3 = b^3 + c^3. \tag{6.11}$$

As usual we let

$$s(q) := \frac{c(q)}{a(q)}, \qquad s^*(q) := \frac{b(q)}{a(q)}. \tag{6.12}$$

Then, from (6.9)–(6.12) we have

$$1 + 2s(q^3) = 1 + 2\frac{c(q^3)}{a(q^3)} = \frac{a(q)}{a(q^3)}, \tag{6.13}$$

$$1 + 2s^*(q) = 1 + 2\frac{b(q)}{a(q)} = 3\frac{a(q^3)}{a(q)}, \tag{6.14}$$

and

$$s^3 + (s^*)^3 = 1. \tag{6.15}$$

Now, from (6.7), (6.13), (6.14) we have

$$m_{3,3}(r) = \left[1 + 2s(q^3)\right]^2, \tag{6.16}$$

$$\left[1 + 2s(q^3)\right]\left[1 + 2s^*(q)\right] = 3,$$

and, by (3.18), (6.16) we have

$$\alpha_3(9r) = [1 + 2S(9r)]^2 - \frac{4\sqrt{3r}}{3}S(9r)(1 + S(9r)); \tag{6.17}$$

where

$$S(r) = s(q), \qquad q = \exp(-2\pi\sqrt{r}/\sqrt{3}). \tag{6.18}$$

From (3.19), (3.20) we know that

$$\alpha_3(1/3) = \frac{1}{3}, \tag{6.19}$$

$$m_{3,3}(1/3) = p = 3, \tag{6.20}$$

and so

$$S(3) = \frac{\sqrt{3}-1}{2}. \tag{6.21}$$

By letting,

$$\alpha_n := \alpha\left(9^n \cdot \frac{1}{3}\right), \tag{6.22}$$

$$s_n := S\left(9^n \cdot \frac{1}{3}\right) = s\left(\exp(-2\pi 3^{n-1})\right), \tag{6.23}$$

$$s_n^* := S^*\left(9^n \cdot \frac{1}{3}\right), \tag{6.24}$$

we have the following

Theorem 6.1 *thm6.1 (Borwein and Borwein [6, Iteration 3.2, p. 699]) Define sequences $\{\alpha_n\}$, $\{s_n\}$, and $\{s_n^*\}$ by*

$$\alpha_0 := \alpha(1/3) = \frac{1}{3}; \tag{6.25}$$

$$s_1 : = S(3) = \frac{\sqrt{3}-1}{2}; \tag{6.26}$$

$$(s_n)^3 + (s_n^*)^3 = 1; \tag{6.27}$$

$$(1+2s_n)(1+2s_{n-1}^*) = 3; \tag{6.28}$$

$$\alpha_n = (1+2s_n)^2\alpha_{n-1} - 4\cdot 3^{n-2}(1+s_n)s_n. \tag{6.29}$$

Then α_n converges cubically to $\frac{1}{\pi}$.

7 The Septic

In this section we examine the case $p = 7$ and obtain an iteration that converges to $1/\pi$ to 7-th order. The proofs of some of the results of this section have been omitted. They will appear in a later version of this *organic* paper. Our idea is to mimic the cubic iteration of Section 6. The quadratic form in the cubic case is $n^2 + nm + m^2$, which has discriminant -3 and class number $h = h(-3) = 1$. The case $p = 7$ is the next case to consider since the next odd, negative discriminant with class number 1 is -7. The corresponding quadratic form is $2n^2 + 3mn + 2n^2$. See [9] for a treatment of the classical theory of binary quadratic forms. We have

$$A_7(q) = [a_7(q)]^2 \tag{7.1}$$

where

$$a_7(q) = 1 + 2\sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{q^n}{1-q^n} \tag{7.2}$$

$$= \sum_{m,n=-\infty}^{\infty} q^{2n^2+3mn+2m^2}. \tag{7.3}$$

Also,

$$\begin{aligned} [a_7(q)]^3 &= \frac{\eta^7(\tau)}{\eta(7\tau)} + 49\frac{\eta^7(7\tau)}{\eta(\tau)} + 13\eta^3(\tau)\eta^3(7\tau) \\ &= B_7 + 49C_7 + 13\sqrt{B_7C_7} \\ &= \left[\sqrt{B_7} + 7\sqrt{C_7}\right]^2 - \sqrt{B_7C_7}. \end{aligned} \tag{7.4}$$

Equation (7.1) with (7.2), and (7.4) are contained in Entry 5(i) of Chapter 21 of Ramanujan's second notebook [13], [1, p. 467]. Equation (7.3) is analogous to (6.8) but does not appear in Ramanujan's notebooks. We expect a result like (7.3) to hold since the class number $h(-7) = 1$. A result of Legendre's [11, Equation (1), p. 97] gives an equivalent formulation in terms of L-series:

$$2\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{m=1}^{\infty} \left(\frac{m}{7}\right) \frac{1}{m^s} = \sum_{(m,n)\neq(0,0)} \frac{1}{(2n^2+3mn+2m^2)^s}. \tag{7.5}$$

We define the following functions (analogous to the cubic case):

$$a := a(q) = \sum_{m,n=-\infty}^{\infty} q^{2n^2+3mn+2m^2}, \tag{7.6}$$

and for $j = 1, 2, 3$

$$b_j := \sum_{m,n=-\infty}^{\infty} \omega^{j(m-n)} q^{2n^2+3mn+2m^2}, \qquad (\omega = \exp(2\pi i/7)) \tag{7.7}$$

$$c_j := \sum_{m,n=-\infty}^{\infty} q^{2(n+\frac{j}{7})^2+3(m+\frac{j}{7})(n+\frac{j}{7})+2(m+\frac{j}{7})^2}. \tag{7.8}$$

In a later version of this paper we will examine these functions under the action of the congruence subgroup, $\Gamma_0(7)$.

The proof of the following is analogous to the cubic case [7, Lemma 2.1, p. 36]:

$$a(q^7) = \frac{1}{7}(a(q) + 2(b_1(q) + b_2(q) + b_3(q)), \tag{7.9}$$

$$c_1(q^7) = \frac{1}{7}(a + (\omega^2 + \omega^5)b_1 + (\omega^3 + \omega^4)b_2 + (\omega + \omega^6)b_3) \tag{7.10}$$

$$c_2(q^7) = \frac{1}{7}(a + (\omega^3 + \omega^4)b_1 + (\omega + \omega^6)b_2 + (\omega^2 + \omega^5)b_3) \tag{7.11}$$

$$c_3(q^7) = \frac{1}{7}(a + (\omega + \omega^6)b_1 + (\omega^2 + \omega^5)b_2 + (\omega^3 + \omega^4)b_3). \tag{7.12}$$

From (7.9)–(7.12) we easily find that

$$c_1(q^7) + c_2(q^7) + c_3(q^7) = \frac{1}{7}(3a - b_1 - b_2 - b_3) \tag{7.13}$$

$$= \frac{1}{2}(a(q) - a(q^7)). \tag{7.14}$$

For $j = 1, 2, 3$ we define

$$s_j := \frac{c_j}{a}, \tag{7.15}$$

$$s_j^* := \frac{b_j}{a}, \tag{7.16}$$

and

$$\sigma := s_1 + s_2 + s_3, \tag{7.17}$$

$$\sigma^* := s_1^* + s_2^* + s_3^*. \tag{7.18}$$

From (3.14), (7.1) we have

$$m_{7,7}(r) = \left(\frac{a_7(q)}{a_7(q^7)}\right)^2. \tag{7.19}$$

Throughout this section q and r are related by

$$q = \exp(-2\pi\sqrt{r}/\sqrt{7}). \tag{7.20}$$

If we write the s_j as functions of r then by using the multidimensional analogue [?] of (3.3) it can be shown that

$$s_j\left(\frac{1}{r}\right) = s_j^*(r), \tag{7.21}$$

for $j = 1, 2, 3$.

By (7.14), (7.19) we have

$$m_{7,7}(r) = \left[1 + 2\sigma(q^7)\right]^2, \tag{7.22}$$

$$\left[1 + 2\sigma(q^7)\right][1 + 2\sigma^*(q)] = 7. \tag{7.23}$$

As noted in Section 4.3 the main problem of constructing a 7-th order iteration is to find a relationship between $m_{7,7}(49r)$ and $m_{7,7}(r)$. In view of (7.22) and (7.23), we would like to find a relationship between $\sigma(q)$ and $\sigma^*(q)$. Instead we get each of the s_j^* in terms of s_1, s_2, s_3. This is achieved by solving a certain cubic and taking 7-th roots.

We now give some details of this construction. Let

$$\alpha = (s_1^*)^3\, s_2^*, \qquad \beta = (s_2^*)^3\, s_3^*, \qquad \gamma = (s_3^*)^3\, s_1^*. \tag{7.24}$$

Now we define the following functions:

$$g_1 = s_1 s_2 s_3, \tag{7.25}$$

$$g_2 = s_1^3 s_2 + s_2^3 s_3 + s_3^3 s_1, \tag{7.26}$$

$$g_3 = 1 - \frac{10}{7} g_1 + \frac{1}{7} g_2 \tag{7.27}$$

$$g_4 = 3 - \frac{51}{7} g_1 + \frac{10}{7} g_2. \tag{7.28}$$

Then by using the theory of modular functions it can be shown that

$$\alpha + \beta + \gamma = g_4, \tag{7.29}$$

$$(\alpha\beta + \beta\gamma + \gamma\alpha) = g_3(2g_4 - 3g_3), \tag{7.30}$$

$$\alpha\beta\gamma = (s_1^* s_2^* s_3^*)^4 = g_3^4. \tag{7.31}$$

Also,

$$\delta := s_1^* s_2^* s_3^* = g_3. \tag{7.32}$$

Our functions g_1 and g_2 are related. We find that

$$g_2 = \frac{7g_1 + 6 + \sqrt{49 - (g_1 - 1)(27g_1 - 13)}}{2}. \tag{7.33}$$

Hence α, β, γ are the roots of the following cubic equation

$$x^3 - g_4 x^2 + x\, g_3(2g_4 - 3g_3) - g_3^4 = 0. \tag{7.34}$$

Now observe that

$$\frac{\alpha^3\gamma}{\delta^3} = \frac{({s^*}_1^3 {s^*}_2)^3 ({s^*}_3^3 {s^*}_1)}{({s^*}_1 {s^*}_2 {s^*}_3)^3} = {s_1^*}^7. \tag{7.35}$$

Similarly we find that

$$s_1^* = \left(\frac{\alpha^3\gamma}{\delta^3}\right)^{\frac{1}{7}}, \quad s_2^* = \left(\frac{\beta^3\alpha}{\delta^3}\right)^{\frac{1}{7}}, \quad s_3^* = \left(\frac{\gamma^3\beta}{\delta^3}\right)^{\frac{1}{7}}. \tag{7.36}$$

Thus after solving a certain cubic, and taking 7-th roots we may obtain each of the s_j^* in terms of s_1, s_2, and s_3.

From (3.18) we have

$$\alpha_7(49r) = \alpha_7(r)\, m_{7,7}(r) + \frac{\sqrt{7r}}{3}(1 - m_{7,7}(r)). \tag{7.37}$$

Finally, to construct an septic iteration, we need some initial values. Two obvious candidates are $r_0 = 1,\ 1/7$. From (3.10) we have

$$\alpha_7(1) = \frac{4}{3\sqrt{7}}. \tag{7.38}$$

We need to find the values of the s_j for $r = 1$. Fortunately, by (7.21), these coincide with the values of the s_j^*. So when $r = 1$, $g_1 = g_3$ and $g_2 = g_4$. Thus we can in this case solve equations (7.27) and (7.33) to obtain

$$g_1(1) = \frac{13}{27}, \qquad g_2(1) = \frac{32}{27}. \tag{7.39}$$

Thus via (7.34) we find that $\alpha(1)$, $\beta(1)$, $\gamma(1)$ are roots of the cubic

$$27^4\, x^3 - 27^3 \cdot 32\, x^2 + 325 \cdot 27^2\, x - 13^4 = 0. \tag{7.40}$$

Then after solving this cubic we can find the initial values $s_j(1) = s_j^*(1)$ using (7.36). To obtain the septic iteration we need only consider $r = 49^n$ in our identities and equations to obtain the following theorem.

Theorem 7.1 *thm3 Define sequences* $\{\alpha_n\}$, $\{\mathbf{s}_n\}$, $\{\mathbf{s}_n^*\}$, *and* $\{m_n\}$ *by*

$$\alpha_0 := \frac{4}{3\sqrt{7}}; \tag{7.41}$$

$$\mathbf{s}_0 := \left(\frac{27}{13}\right)^{3/7} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \tag{7.42}$$

where

$$y_1 = \left(x_1^3 x_3\right)^{\frac{1}{7}}, \qquad y_2 = \left(x_2^3 x_1\right)^{\frac{1}{7}}, \qquad y_3 = \left(x_3^3 x_2\right)^{\frac{1}{7}}, \tag{7.43}$$

and $x_2 < x_1 < x_3$ *are the roots of the cubic equation*

$$27^4\, x^3 - 27^3 \cdot 32\, x^2 + 325 \cdot 27^2\, x - 13^4 = 0; \tag{7.44}$$

$$\mathbf{s}_n^* = \begin{pmatrix} (\alpha^3\gamma/g_3^3)^{1/7} \\ (\beta^3\alpha/g_3^3)^{1/7} \\ (\gamma^3\beta/g_3^3)^{1/7} \end{pmatrix}, \tag{7.45}$$

where $\beta < \alpha < \gamma$ *are the roots of the cubic*

$$x^3 - g_4 x^2 + x\, g_3(2g_4 - 3g_3) - g_3^4 = 0; \tag{7.46}$$

where

$$g_3 = 1 - \frac{10}{7} g_1 + \frac{1}{7} g_2, \tag{7.47}$$

$$g_4 = 3 - \frac{51}{7} g_1 + \frac{10}{7} g_2, \tag{7.48}$$

and where

$$g_1 = s_{1,n}\, s_{2,n}\, s_{3,n}, \tag{7.49}$$

$$g_2 = s_{1,n}^3 s_{2,n} + s_{2,n}^3 s_{3,n} + s_{3,n}^3 s_{1,n},; \tag{7.50}$$

$$m_n = \frac{49}{(1 + 2\mathbf{s}_n^* \cdot \mathbf{1})^2}, \qquad \textit{(where } \mathbf{1} = (1,1,1)^t\textit{);} \tag{7.51}$$

$$\mathbf{s}_n = \frac{1}{7} \sqrt{m_{n-1}} \left[M\, \mathbf{s}_{n-1}^* + \mathbf{1} \right] \tag{7.52}$$

where

$$M = \left(2 \cos\left(\frac{4\pi i j}{7} \right) \right)_{1 \le i,j \le 3}; \tag{7.53}$$

and

$$\alpha_n = m_{n-1}\, \alpha_{n-1} + \sqrt{7}\ \frac{7^{n-1}}{3} \left(1 - m_{n-1} \right). \tag{7.54}$$

Then α_n *converges septically to* $\frac{1}{\pi}$.

8 Nonic Iterations

In this section we explore nonic (ninth order) iterations. Of course, we may construct a ninth order iteration by iterating the cubic. We have found a nonic iteration which does not seem to come from the cubic. Our nonic iteration is based on a symbolic discovery that $A_9(q)$ is a nice eta-product:

$$A_9(q) = \frac{\eta^{10}(3\tau)}{\eta^3(\tau)\eta^3(9\tau)}, \qquad (q = \exp(2\pi i \tau). \tag{8.1}$$

This identity can be proved easily from results in Ramanujan's notebook. From (3.15), (3.16) we have

(8.2)
$$
\begin{aligned}
A_9(q) &= 1 + 3\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 27\sum_{n=1}^{\infty} \frac{nq^{9n}}{1-q^{9n}},\\
&= \frac{\eta^6(3\tau)}{\eta^2(\tau)\eta^2(9\tau)} \left\{\eta^6(\tau) + 9\eta^3(\tau)\eta^3(9\tau) + 27\eta^6(9\tau)\right\}^{1/3},\\
&\qquad \text{(by [1, Entry 7(i), Chapter 21, p. 475])},\\
&= \frac{\eta^{10}(3\tau)}{\eta^3(\tau)\eta^3(9\tau)},
\end{aligned}
$$

by [1, Entry 1(iv), Chapter 20, p. 345]. In the last step we have used the classical cubic modular equation for the eta-function:

$$
\left(1 + 9\frac{\eta^3(9\tau)}{\eta^3(\tau)}\right)^3 = 1 + 27\left(\frac{\eta(3\tau)}{\eta(\tau)}\right)^{12}. \tag{8.3}
$$

See [7, Corollary 2.5]. Using (6.5), (6.6) $A_9(q)$ may be written terms of either $b(q)$ or $c(q)$ (defined in (6.2), (6.3)), which were involved in the cubic iteration. We have

$$
A_9(q) = \frac{1}{9}\,\frac{c^3(q)}{c(q^3)} = \frac{b^3(q^3)}{b(q)}. \tag{8.4}
$$

This time we let

$$
s(q) = \frac{c(q^3)}{a(q^3)}, \qquad s^*(q) = \frac{b(q^3)}{a(q^3)}, \tag{8.5}
$$

where $a(q)$, $b(q)$, $c(q)$ are defined in (6.1)–(6.3). We define the following functions:

$$
T(q) := 1 + 2s^*(q), \tag{8.6}
$$

$$
U(q) := \left[9s^*(q)\,(1 + s^*(q) + (s^*(q))^2)\right]^{1/3}. \tag{8.7}
$$

Using the results of Section 6 it can be shown that

$$
s(q^9) = \frac{(1 - s^*(q))^3}{(T(q) + 2U(q))(T^2(q) + T(q)U(q) + U^2(q))}, \tag{8.8}
$$

$$
m_{9,9}(r) = \frac{27(1 + s(q) + s^2(q))}{T^2(q) + T(q)U(q) + U^2(q)}, \tag{8.9}
$$

where $q = \exp(-2\pi\sqrt{r}/3)$. We can also utilize the initial values for the cubic and obtain a nonic iteration.

Theorem 8.1 *thm8.1 Define sequences $\{\alpha_n\}$, $\{s_n\}$, and $\{s_n^*\}$ by*

$$\alpha_0 := \frac{1}{3}; \tag{8.10}$$

$$s_1^* : = \frac{\sqrt{3}-1}{2}; \tag{8.11}$$

$$s_1 : = (1-(s_1^*)^3)^{1/3}; \tag{8.12}$$

$$\alpha_n = m\,\alpha_{n-1} + 3\cdot 9^{n-2}(1-m), \tag{8.13}$$

where

$$m := 27\frac{(1+s_n+s_n^2)}{t^2+tu+u^2}, \tag{8.14}$$

$$t := 1+2s_n^*, \tag{8.15}$$

$$u := \left[9s_n^*(1+s_n^*+(s_n^*)^2)\right]^{1/3} \tag{8.16}$$

and where

$$s_{n+1} = \frac{(1-s_n^*)^3}{(t+2u)(t^2+tu+u^2)}. \tag{8.17}$$

Then α_n converges nonically to $\frac{1}{\pi}$.

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